GEOMETRICAL SIGNIFICANCE OF ISOTHERMAL CONJUGACY OF A NET OF CURVES.

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INTRODUCTION.

Let

$$(1) \qquad Ddu^2 + 2D'dudv + D''dv^2$$

be the second fundamental differential form of a surface S, and let us consider a region R on this surface which is free from parabolic points so that, for all points in R,

$$(2) \qquad D'^2 - DD'' \neq 0.$$

If D' is equal to zero for all points of R, the curves $u =$ const. and $v =$ const. form a *conjugate* net. If this condition is satisfied, and if besides the ratio $D : D''$ assumes the form of a function of u alone multiplied by a function of v alone, so that

$$(3) \qquad \frac{\partial^2 \log D/D''}{\partial u \partial v} = 0, \qquad D' = 0,$$

the net is said to be *isothermally* conjugate. This name is due to Bianchi,* and was chosen by him because, in all such cases, it is possible to choose new variables

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v)$$

in such a way as to transform (1) into the isothermal form

$$\lambda(\bar{u}, \bar{v})(d\bar{u}^2 + d\bar{v}^2),$$

without changing the conjugate net under consideration.

Bianchi also proved that the property of isothermal conjugacy is of a *projective* character.† That is, if an isothermally conjugate net is subjected to any projective transformation, the resulting net will again be isothermally conjugate. But Bianchi did not furnish any geometric interpretation of the analytic conditions (3) which serve to define such systems. Moreover, although the importance of this notion was becoming more and more apparent, because of a steadily increasing body of theorems which made use of it, no serious attempt seems to have been made to discover its true significance until 1915, when the author of the present paper discovered an algebraic relation, between certain completely interpreted projective in-

* L. Bianchi, "Lezioni di geometria differenziale" (Seconda edizione), Vol. 1, p. 168.
† Ibid. p. 169.

variants, which is characteristic of isothermally conjugate systems.* Thus, in a sense, the problem was solved. But the solution was not altogether satisfying because it lacked simplicity and could not be formulated completely in terms of purely descriptive relations. A year afterward, the late G. M. Green, whose premature death has deprived geometry of one of its most brilliant students, took a long step in advance.† In fact, Green believed that he had settled the matter completely. But he had overlooked an important case in which his geometric criterion fails to distinguish between isothermally conjugate nets and nets of an entirely different kind.

The present paper was written for the purpose of completing the solution of this problem, as nearly as possible in the spirit of Green's method, and making use of Green's notations. I dedicate this paper to his memory.

1. Résumé and Revision of Green's Theory.

Let

$$y^{(k)} = y^{(k)}(u, v), \qquad (k = 1, 2, 3, 4) \tag{4}$$

be the homogeneous coördinates of a point P_y. When the variables, u and v, vary over their ranges, P_y will in general describe a surface S_y. We shall assume that this surface does not degenerate into a curve, and that it is non-developable. If the curves $u =$ const. and $v =$ const. form a conjugate net on S_y, there exists a completely integrable system of differential equations of the form

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \qquad a \neq 0, \\ y_{uv} &= * + b'y_u + c'y_v + d'y, \end{aligned} \tag{5}$$

whose fundamental, linearly independent, solutions are $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$. Conversely, every completely integrable system of form (5) defines a non-developable surface referred to a conjugate net.

The integrability conditions of system (5) teach us that there exists a function p, of u and v, such that‡

$$p_u = b + 2c', \qquad p_v = \frac{2ab' - c - a_v}{a}. \tag{6}$$

Consequently we can make a transformation of the form

$$y = \lambda \bar{y}, \tag{7}$$

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 323. Quoted hereafter as W.

† G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves and Conjugate Nets on a Curved Surface (Second Memoir), Am. Jour. of Math., Vol. 38 (1916), p. 323. Quoted hereafter as Green (Second Memoir).

‡ G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves, etc. (First Memoir), Am. Jour. of Math., Vol. 37 (1915), p. 223. Quoted hereafter as Green (First Memoir).

where λ is subjected to the conditions

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{4}p_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{4}p_v. \tag{8}$$

The resulting system of differential equations has the same form as (5), with the coefficients*

$$\begin{aligned} &A = a, \qquad B = b - \tfrac{1}{2}p_u, \qquad C = c + \frac{a}{2}p_v, \\ &D = d + \tfrac{1}{4}bp_u + \tfrac{1}{4}cp_v - \tfrac{1}{4}p_{uu} + \tfrac{1}{4}ap_{vv} - \tfrac{1}{16}p_u^2 + \tfrac{1}{16}ap_v^2, \\ &B' = b' - \tfrac{1}{4}p_v, \qquad C' = c' - \tfrac{1}{4}p_u, \\ &D' = d' + \tfrac{1}{4}b'p_u + \tfrac{1}{4}c'p_v - \tfrac{1}{4}p_{uv} - \tfrac{1}{16}p_up_v. \end{aligned} \tag{9}$$

These coefficients are *seminvariants* of (5), and the new system is said to be in its *canonical form.* The relations

$$B + 2C' = 0, \qquad 2AB' - C - A_v = 0, \tag{10}$$

which follow from (9), are characteristic of this canonical form.

Any proper transformation of the form

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v), \tag{11}$$

affects only the parametric representation of the conjugate net given by (5), but leaves the net itself unchanged. The *invariants* of the net are those functions of the seminvariants, which remain unchanged by transformations of form (11), except for a factor. The fundamental invariants are†

$$\begin{aligned} &\mathfrak{A} = A, \qquad \mathfrak{B}' = B' - \frac{3}{8}\frac{A_v}{A}, \qquad \mathfrak{C}' = C' + \frac{1}{8}\frac{A_u}{A}, \\ &\mathfrak{D}' = D' + B'C', \qquad \mathfrak{D} = D - (B'A_v - AB'_v) - C'_u + 3(AB'^2 - C'^2); \end{aligned} \tag{12}$$

besides these, the following two, the Laplace-Darboux invariants of the net,‡

$$H = D' + B'C' - B_u, \qquad K = D' + B'C' - C'_v \tag{13}$$

are especially important.

The curves $u = \text{const.}$ and $v = \text{const.}$ of our conjugate net are not asymptotic lines. Therefore, the osculating planes of the two curves of the net, which meet at a point P_y of the surface, determine, as their line of intersection, a line passing through P_y and not in the tangent plane. This line is called the *axis* of P_y, and the totality of all such lines is called the *axis congruence* of the given conjugate system. The developables of the

* Green (First Memoir), p. 224.

† Ibid., p. 226.

‡ Ibid., p. 231–232.

axis congruence correspond to a net of curves on S_y, called the *axis curves*, whose differential equation is*

$$(14)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right) du^2 - \mathfrak{D} du dv - (H + 2b'_u - b_v) dv^2 = 0,$$

where a, b, and b' may be replaced by A, B, B' and where the relations (10) may then be used. The *anti-axis curves* are defined by*

$$(15)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right) du^2 + \mathfrak{D} du dv - (H + 2b'_u - b_v) dv^2 = 0.$$

Their tangents at any point of the surface are the harmonic conjugates of the axis curve tangents with respect to the tangents of the original conjugate system $u =$ const., $v =$ const.

The covariants

$$(16)\qquad \rho = y_u - c'y, \qquad \sigma = y_v - b'y$$

are the variables which determine the Laplace transformations of system (5). The points P_ρ and P_σ are in the plane tangent to S_y at P_y. The locus of P_ρ is the second sheet of the focal surface of the congruence formed by the tangents of the curves $v =$ const. on S_y. P_σ is connected in the same way with the congruence of tangents of the curves $u =$ const. on S_y. The line $P\rho P\sigma$, which moreover corresponds to the axis of P_y by duality, is called the *ray of* P_y. The totality of rays, for all surface points, is called the *ray congruence*, and the curves on S_y which correspond to the developables of the ray congruence, are called its *ray curves*.* The differential equation of the ray curves is

$$(17)\qquad aH du^2 - \mathfrak{D} du dv - K dv^2 = 0.$$

The *anti-ray curves* are related to the ray curves in the same way as the axis curves to the anti-axis curves. Their differential equation is as follows†;

$$(18)\qquad aH du^2 + \mathfrak{D} du dv - K dv^2 = 0.$$

There exists a uniquely determined conjugate net on the surface such that the two tangents of this new net, at any point of the surface, shall separate not only the asymptotic tangents, but also the tangents of the original conjugate system, harmonically. Green has called this system of curves the *associate conjugate net*,‡ and found its differential equation to be

$$(19)\qquad a du^2 - dv^2 = 0,$$

the asymptotic net of S_y being determined by

$$(20)\qquad a du^2 + dv^2 = 0.$$

* W., pp. 314–316 and Green (Second Memoir), pp. 308 and 310.

† W., pp. 317–318 and Green (Second Memoir), p. 309.

‡ Green (Second Memoir), p. 313.

In the case of an isothermally conjugate net, a has the form of a product of a function of u alone by a function of v alone, so that

$$\frac{\partial^2 \log a}{\partial u \partial v} = 0, \qquad a \neq 0. \tag{21}$$

It will then be possible to find a transformation

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

such that the value of a in the transformed differential equations becomes equal to unity. Thus, *if the parametric net is isothermally conjugate, we may assume*

$$a = 1. \tag{22}$$

Let us consider the three quadratics (14), (18), and (19). The Jacobian of (14) and (19) is

$$a\mathfrak{D}du^2 + 2a\left(H - K + \frac{\partial^2 \log a}{\partial u \partial v}\right)dudv + \mathfrak{D}dv^2 = 0; \tag{23}$$

the Jacobian of (18) and (19) is

$$a\mathfrak{D}du^2 + 2a(H - K)dudv + \mathfrak{D}dv^2 = 0, \tag{24}$$

and clearly these Jacobians are equivalent, as quadratics in $du : dv$, if (21) is satisfied. But they are also equivalent if $\mathfrak{D} = 0$, and this is the case which Green failed to consider. In this exceptional case the axis curves and ray curves are so related to the parametric conjugate system that at every surface point the tangents belonging to the latter are separated harmonically by the tangents of each of the former nets, unless still other invariants vanishing cause one or both of these nets to become indeterminate. On account of these properties, let us call such conjugate nets, characterized by the condition $\mathfrak{D} = 0$, *harmonic conjugate nets.*

We have proved the following theorem.

THEOREM 1. *A conjugate net whose axis tangents, anti-ray tangents, and associate conjugate tangents, form three pairs of an involution at every point of the net, is either isothermally conjugate, or harmonic, or both.*

In this theorem, the axis tangents and anti-ray tangents may be replaced simultaneously by the anti-axis tangents and ray tangents, respectively.

Since it is our purpose to characterize isothermally conjugate nets completely by geometric properties, we must now search for properties of such nets which they do *not* share with harmonic conjugate nets. In most cases the following theorem will enable us to distinguish between harmonic and isothermally conjugate nets.

THEOREM 2. *The involution, mentioned in Theorem 1, has the parametric*

conjugate tangents as its double lines, if and only if the original net is harmonic. Therefore, the given net is isothermally conjugate, and not harmonic, if the three pairs of tangents mentioned in Theorem 1 are pairs of an involution, and if, besides, the double lines of this involution do not coincide with the parametric tangents.

If, however, the double elements of this involution *do* coincide with the parametric tangents, we can only conclude that the given net is harmonic. It may or may not be isothermally conjugate, at the same time. Thus our geometric criterion fails to distinguish between nets which are both isothermally conjugate and harmonic, and those which are merely harmonic.

Green* has shown that the associate conjugate net of an isothermally conjugate net is also isothermally conjugate, and vice versa, a theorem which we shall generalize in the next section. We may, therefore, apply theorems 1 and 2 to the associate conjugate net, obtaining the following result.

THEOREM 3. *If a conjugate net is isothermally conjugate, the associate conjugate net is also isothermally conjugate and vice versa. Consequently, the associate axis tangents, the associate anti-ray tangents, and the conjugate tangents of the original net, at any point of the net, will form three pairs of an involution. The double lines of this second involution will coincide with the associate conjugate tangents if and only if the associate conjugate net is harmonic.*

The associate axis tangents, etc., mentioned in this theorem, are related to the associate conjugate system in the same manner as the axis tangents, etc. are to the original system. By combining theorems 1, 2, 3, we obtain the following criterion.

THEOREM 4. *For an isothermally conjugate net both of the involutions, mentioned in theorems 1 and 3 exist. Conversely, if both of these involutions exist for a conjugate net, we can conclude that the net is isothermally conjugate unless both the original net and its associate net are harmonic.*

2. PENCILS OF CONJUGATE NETS ON A SURFACE.

Theorem 4 seems to be the most comprehensive criterion which can be obtained without introducing something essentially new into the discussion, but it does not solve the problem completely. For, it does not enable us to distinguish geometrically between conjugate nets which are harmonic, possess a harmonic associate net, and are besides isothermally conjugate, and conjugate nets which possess merely the first two of these properties. In order to solve our problem completely we introduce a new notion, that of a *pencil of conjugate systems,* a notion which we shall introduce at present only in connection with our special problem but which seems to be one of considerable general importance.

* Green (Second Memoir), p. 324.

Let us assume that the given conjugate system is isothermally conjugate, let the independent variables be chosen so that $a = 1$, and let the equations (5) be taken in their canonical form. Then we shall have

$$
\begin{aligned}
y_{uu} &= y_{vv} + By_u + Cy_v + Dy, \qquad a = A = 1, \\
y_{uv} &= \quad * \quad + B'y_u + C'y_v + D'y,
\end{aligned} \tag{25}
$$

where, on account of (10),

$$B = -2C', \qquad C = 2B'. \tag{26}$$

The differential equations of the original conjugate system will be $dudv = 0$, that of the associate system will be $du^2 - dv^2 = 0$, and that of the asymptotic lines will be $du^2 + dv^2 = 0$. The differential equation

$$\alpha du^2 + 2\beta dudv + \gamma dv^2 = 0 \tag{27}$$

will determine a conjugate net if and only if

$$\alpha + \gamma = 0,$$

a condition obtained by equating to zero the harmonic invariant of (27) and $du^2 + dv^2 = 0$, the differential equation of the asymptotic lines. The tangents, at any point, of the curves of such a conjugate net will divide the corresponding tangents of the original conjugate net in a *constant* cross-ratio, if and only if the ratio of α to β is a constant. Consequently, *the differential equation*

$$(du + kdv)(kdu + dv) = 0, \tag{28}$$

where k is an arbitrary constant, will determine a one-parameter family of conjugate nets each of which has the property that, at every point, the two tangents which belong to it determine a constant cross-ratio with those which belong to the original net.

We shall speak of the one-parameter family of conjugate nets, determined in this way by a given one, as a *pencil of conjugate nets*. There is one such net for every value, real or complex, of the constant k, but it is clear from (28) that the same net will correspond to two values of k which are negative reciprocals of each other. The net which corresponds to $k = 0$ or $k = \infty$ is the original net, and that which corresponds to the values $k = \pm 1$ is the associate net. For $k = \pm i$ the two factors of (28) become identical with each other and with one of the factors of $du^2 + dv^2$; the net degenerates into one of the families of asymptotic lines counted twice and therefore is not, properly speaking, a net at all. By a *proper* net of the pencil we mean any one of its nets excepting the two just mentioned which correspond to $k = i$ and $k = -i$. Of course every proper net of a pencil may be regarded as determining, in its turn, a pencil of nets.

But it follows at once, from the definition of a pencil, that all of these pencils coincide with each other and with the original pencil, and further that the nets of a pencil may be arranged in pairs associate to each other.

In order to study the properties of an individual net of the pencil, we introduce the variabels

$$\bar{u} = u - kv, \qquad \bar{v} = ku + v \tag{29}$$

into (25) in place of u and v. These variables will be independent if $1 + k^2$ is different from zero. We shall assume

$$1 + k^2 \neq 0, \tag{30}$$

a hypothesis which excludes from consideration only the improper conjugate systems formed by each of the two sets of asymptotic lines. We find

$$\begin{aligned} y_u &= y_{\bar{u}} + ky_{\bar{v}}, \qquad y_v = -ky_{\bar{u}} + y_{\bar{v}}, \\ y_{uu} &= y_{\bar{u}\bar{u}} + 2ky_{\bar{u}\bar{v}} + k^2 y_{\bar{v}\bar{v}}, \\ y_{uv} &= -ky_{\bar{u}\bar{u}} + (1 - k^2)y_{\bar{u}\bar{v}} + ky_{\bar{v}\bar{v}}, \\ y_{vv} &= k^2 y_{\bar{u}\bar{u}} - 2ky_{\bar{u}\bar{v}} + y_{\bar{v}\bar{v}}. \end{aligned} \tag{31}$$

Substitution of these values into equations (25) gives

$$\begin{aligned} (1 - k^2)y_{\bar{u}\bar{u}} + 4ky_{\bar{u}\bar{v}} - (1 - k^2)y_{\bar{v}\bar{v}} &= B(y_{\bar{u}} + ky_{\bar{v}}) + C(-ky_{\bar{u}} + y_{\bar{v}}) + Dy, \\ -ky_{\bar{u}\bar{u}} + (1 - k^2)y_{\bar{u}\bar{v}} + ky_{\bar{v}\bar{v}} &= B'(y_{\bar{u}} + ky_{\bar{v}}) + C'(-ky_{\bar{u}} + y_{\bar{v}}) + D'y, \end{aligned}$$

whence

$$\begin{aligned} (1 + k^2)^2(y_{\bar{u}\bar{u}} - y_{\bar{v}\bar{v}}) = {} & (1 - k^2)[(B - kC)y_{\bar{u}} + (kB + C)y_{\bar{v}} + Dy] \\ & - 4k[(B' - kC')y_{\bar{u}} + (kB' + C')y_{\bar{v}} + D'y], \\ (1 + k^2)^2 y_{\bar{u}\bar{v}} = {} & k[(B - kC)y_{\bar{u}} + (kB + C)y_{\bar{v}} + Dy] \\ & + (1 - k^2)[(B' - kC')y_{\bar{u}} + (kB' + C')y_{\bar{v}} + D'y]. \end{aligned} \tag{32}$$

These equations show, in the first place, that the new conjugate net, $\bar{u} =$ const., $\bar{v} =$ const., is isothermally conjugate, giving

THEOREM 5. *An isothermally conjugate net determines a pencil, all of whose proper nets are isothermally conjugate.*

This theorem includes, as a special case, Green's theorem that the associate net of an isothermally conjugate net is also isothermally conjugate. But we may draw a still farther reaching conclusion, by remembering that the same pencil of nets is determined if we start from any one of its proper nets in place of the one actually used. We then obtain the following result.

THEOREM 6. *If a pencil of conjugate nets contains one isothermally conjugate net, then all proper nets of the pencil are isothermally conjugate.*

We may reduce (32) to the form (25) by dividing by $(1 + k^2)^2$. If we denote the corresponding coefficients by A_k, B_k, C_k, etc., we find

$$(33)\qquad \begin{aligned}(1+k^2)^2B_k &= (1-k^2)(B-kC)-4k(B'-kC'),\\ (1+k^2)^2C_k &= (1-k^2)(kB+C)-4k(kB'+C'),\\ (1+k^2)^2D_k &= (1-k^2)D-4kD',\\ (1+k^2)^2B'_k &= k(B-kC)+(1-k^2)(B'-kC'),\\ (1+k^2)^2C'_k &= k(kB+C)+(1-k^2)(kB'+C'),\\ (1+k^2)^2D'_k &= kD+(1-k^2)D',\end{aligned}$$

and

$$(34)\qquad A_k=1,\qquad B_k=-2C'_k,\qquad C_k=2B'_k.$$

The relation $A_k = 1$ is equivalent to Theorem 5. The other two relations in (34) may be verified by means of (26) and (33). They show that our transformed system of differential equations is in its canonical form.

The invariant $\mathfrak{D}$, whose vanishing characterizes the original conjugate system as a harmonic one, reduces to

$$(35)\qquad \mathfrak{D} = D + B'_v - C'_u + 3(B'^2 - C'^2),$$

since we are assuming $A = 1$. Let us denote by $\mathfrak{D}_k$ the corresponding invariant for any conjugate system of the pencil, so that

$$(36)\qquad \mathfrak{D}_k = D_k + (B'_k)_{\bar v} - (C'_k)_{\bar u} + 3(B'_k)^2 - 3(C'_k)^2.$$

From (33) and (26) we find

$$(37)\qquad \begin{aligned}(1+k^2)^2B'_k &= (1-3k^2)B' + (k^3-3k)C',\\ (1+k^2)^2C'_k &= -(k^3-3k)B' + (1-3k^2)C',\\ (1+k^2)^2D_k &= (1-k^2)D - 4kD'.\end{aligned}$$

If θ is any function of u and v, we find from (31),

$$(38)\qquad \theta_{\bar u} = \frac{1}{1+k^2}(\theta_u - k\theta_v),\qquad \theta_{\bar v} = \frac{1}{1+k^2}(k\theta_u + \theta_v).$$

Consequently we obtain the formulæ

$$\begin{aligned}(1+k^2)^3(C'_k)_{\bar u} &= -(k^3-3k)(B'_u - kB'_v) + (1-3k^2)(C'_u - kC'_v),\\ (1+k^2)^3(B'_k)_{\bar v} &= (1-3k^2)(kB'_u + B'_v) + (k^3-3k)(kC'_u + C'_v),\end{aligned}$$

whence

$$(39)\qquad \begin{aligned}(1+k^2)^4\mathfrak{D}_k = (1+k^2)^2[(1-k^2)D - 4kD' - 2k(B'_u + C'_v)&\\ + (1-k^2)(B'_v - C'_u)]&\\ + 3(1-k^2)(1-14k^2+k^4)(B'^2 - C'^2)&\\ + 12k(1-3k^2)(k^2-3)B'C'.&\end{aligned}$$

For $k = 0$, $\mathfrak{D}_k$ reduces to (35), and for $k = 1$ to

$$(40)\qquad \mathfrak{D}_1 = -D' - \tfrac{1}{2}(B'_u + C'_v) + 3B'C'.$$

Let us assume that $\mathfrak{D}$ and $\mathfrak{D}_1$ are both equal to zero, so that both the original net and its associate are harmonic, besides being isothermally conjugate. Then $\mathfrak{D}_k$ reduces to the value given by

$$(41) \quad (1 + k^2)^4 \mathfrak{D}_k = - 48k(1 - k^2)[k(B'^2 - C'^2) + (1 - k^2)B'C'].$$

If the ratio $B' : C'$ is not a constant, $\mathfrak{D}_k$ can not be equal to zero, for all values of u and v, unless either $k = 0$ or $k = \pm 1$, and these values of k correspond to the original conjugate system and its associate. If the ratio $B' : C'$ *is* a constant which is finite, different from zero or unity, we obtain two values of k, negative reciprocals of each other, and different from 0, ∞, $+ 1$, or $- 1$, by equating to zero the bracketed expression in (41). Thus, there may exist a third net of the pencil, besides the original net and its associate, for which $\mathfrak{D}_k$ is equal to zero. But if $\mathfrak{D}_k$ is equal to zero for more than three distinct nets of the pencil, $\mathfrak{D}_k$ will be equal to zero for all values of k, and B' and C' must vanish. In this case the differential equations of the net reduce to

$$(42) \qquad y_{uu} = y_{vv}, \qquad y_{uv} = 0.$$

Nets of this sort may be described in very simple terms. From equations (42), we conclude

$$y = U(u) + V(v), \qquad U'' = V'' = \tfrac{1}{2}a_1,$$

where $U(u)$ and $V(v)$ are functions of the single variables indicated, and where a_1 is an arbitrary constant. But these equations furnish the following completely integrated expression for y;

$$y = a_1(u^2 + v^2) + a_2 u + a_3 v + a_4,$$

where a_1, a_2, a_3, a_4 are arbitrary constants. The homogeneous parametric equations of such a net may, therefore, be written in the form

$$y_1 = u^2 + v^2, \qquad y_2 = u, \qquad y_3 = v, \qquad y_4 = 1,$$

whence

$$y_1 y_4 - y_2^2 - y_3^2 = 0, \qquad y_2 - u y_4 = 0, \qquad y_3 - v y_4 = 0.$$

Therefore *the sustaining surface of such a net is a quadric. Each of the two component one-parameter families of the net is composed of plane curves (conics), whose planes form a pencil. The axes of these two pencils are conjugate tangents of the quadric surface at one of its points.*

A net with these properties shall be called an *isothermally conjugate quadratic net.* Making use of this terminology we have the following result.

THEOREM 7. *A pencil of isothermally conjugate nets which contains more than three distinct proper harmonic nets is composed entirely of isothermally conjugate quadratic nets.*

We are now in a position to obtain a geometric test for isothermal conjugacy which will be effective in those cases in which theorems 1–4 do not suffice. If a net is isothermally conjugate, every net of its pencil has the property described in theorem 1. If besides, more than three, and

therefore all, of these nets are harmonic, it is an isothermally conjugate quadratic net. Leaving aside this case, we see that the isothermal conjugacy of a net is assured if the property of theorem 1 holds for all of the nets of the pencil and if besides at least one of these nets is known to be non-harmonic.

We may formulate our resulting criterion in the following two theorems.

THEOREM 8. *An isothermally conjugate net possesses the following properties. At every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Moreover, all of the conjugate nets of the pencil, which is determined by the original net, possess this same property, and no more than three of these nets will be, at the same time, harmonic except in the case of an isothermally conjugate quadratic net.*

THEOREM 9. *Conversely: let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property, and assume that at least one of the nets of this pencil is not harmonic. Then the original net is isothermally conjugate. If, however, all of the nets of the pencil are harmonic, the original net is an isothermally conjugate net, if and only if it is an isothermally conjugate quadratic net.*

Theorems 8 and 9 together constitute a set of necessary and sufficient conditions for isothermal conjugacy, and these conditions are expressed in purely geometric form. For, according to theorem 2, the question whether a conjugate net is, or is not harmonic, may be decided by examining the double lines of the corresponding involution.

THE UNIVERSITY OF CHICAGO,
May 4, 1920.

NOTE ADDED OCTOBER 6, 1920.

This criterion may be simplified. I have found recently that, if all of the conjugate nets of a pencil are harmonic, they must also be isothermally conjugate. This remark enables us to replace Theorem 9 by

THEOREM 10. *Conversely, let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property. Then the original net is isothermally conjugate.*

I have also found a second characteristic property of isothermally conjugate nets, which admits of a far simpler statement than that described in theorems 8 and 10. But the detailed presentation of these matters must be left for a future occasion.